Elasto-plastic stress and strain behaviour at notch roots under monotonic and cyclic loadings

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Abstract: Notch deformation behaviour under monotonic and cyclic loading conditions was investigated using circumferentially notched round bar and double-notched flat plate geometries, each with two different notch concentration factors. Notch strains for the double-notched plate geometry were measured with the use of miniature strain gauges bonded to specimens made of a vanadium-based microalloyed steel. Elastic as well as elasto-plastic finite element analyses of the two geometries were performed. Notch root strains and stresses were predicted by employing the linear rule, Neuber’s rule and Glinka’s rule relationships under both monotonic and cyclic loading conditions. The predicted results are compared with those from elastic–plastic finite element analyses and strain gauge measurements. Effects of notch constraint and the material stress–strain curve on the notch root stress and strain predictions are also discussed.

Keywords: notch deformation, monotonic loading, cyclic loading, notch strain, notch stress, microalloyed steel

NOTATION

\( \alpha \) crack length or notch depth
\( C_p \) plastic zone correction factor
\( \varepsilon \) nominal strain
\( \Delta \varepsilon \) nominal strain range
\( E, E' \) monotonic, cyclic modulus of elasticity
\( E^*, E^{**} \) monotonic, cyclic modulus of elasticity for plane strain conditions
\( F \) dimensionless geometry correction factor for stress intensity factor
\( K, K' \) monotonic, cyclic strength coefficient
\( K^*, K^{**} \) monotonic, cyclic strength coefficient for plane strain conditions
\( K_t \) elastic stress concentration factor
\( K_e \) ratio of the maximum strain at the notch root to the nominal strain
\( K_o \) ratio of the maximum stress at the notch root to the nominal stress
\( n, n' \) monotonic, cyclic strain-hardening exponent
\( n^*, n^{**} \) monotonic, cyclic strain-hardening exponent for plane strain conditions
\( r \) notch radius
\( r_p \) plastic zone size
\( \Delta r_p \) increment of the plastic zone size
\( S \) nominal stress
\( S_y, S'_y \) monotonic, cyclic yield strength
\( \alpha \) notch constraint index defined by \( \varepsilon_2/\varepsilon_1 \)
\( \beta = \sigma_2/\sigma_1 \)
\( \varepsilon \) notch root strain
\( \Delta \varepsilon \) notch root strain range
\( \varepsilon_1, \varepsilon_2 \) first, second principal strain at the notch root
\( \varepsilon_{a1}, \varepsilon_{a2} \) first, second principal strain amplitude at the notch root
\( \varepsilon_y^* \) notch root strain in the load direction for plane strain conditions
\( (\varepsilon_y^*)_p \) plastic component of notch root strain in the load direction for plane strain conditions
\( \nu \) Poisson’s ratio
\( \rho \) notch tip radius
\( \sigma \) notch root stress
\( \sigma_1, \sigma_2 \) first, second principal stress at the notch root
\( \sigma_x^*, \sigma_y^* \) notch root stress in the transverse direction
\( \sigma_x \) notch root stress in the load direction for plane strain conditions

1 INTRODUCTION

Many engineering components contain geometrical discontinuities, such as shoulders, keyways, oil holes and grooves,
generally termed notches. When a notched component is loaded, local stress and strain concentrations are generated in the notch area. The stresses often exceed the yield limit of the material in a small region around the notch root, even for relatively low nominally elastic stresses. The local stress and strain concentrations do not usually impair the static strength of a component made from a ductile material, even though plastic deformation takes place at the notch root. When a notched component is subjected to cyclic loading, however, cyclic plastic deformation in the area of stress and strain concentrations can severely reduce service life. The cyclic inelastic strains may cause nucleation of cracks in these highly stressed regions and their subsequent growth could lead to component fracture.

The widely used approaches to notched fatigue behaviour are generally known as the local stress-strain approaches. These approaches are based on relating the crack initiation life at the notch root to the crack initiation life of smooth laboratory specimens. The study of notch behaviour by using the local approach usually includes two steps. The first step is to estimate the local damage using a parameter such as stress, strain or plastic energy density at the notch root. The second step is to predict crack nucleation life based on uniaxial smooth specimen tests, where it is assumed that smooth and notched specimens experience the same number of cycles to failure if they have the same local damage values. Therefore, predicting the local stress-strain behaviour is essential to the understanding of notch fatigue behaviour and of fatigue life prediction.

In this paper, first the commonly used notch stress and strain models are reviewed. Then, a description of the notch geometries used and the experimental strain measurements is provided. Finite element analyses and comparisons with predictions from analytical notch stress and strain rules for both monotonic and cyclic loading conditions are presented. Finally, the results presented are discussed and summarized.

2 NOTCH STRESS–STRAIN MODELS

The well-known and frequently used models for notch stress and strain analyses are the linear rule, Neuber’s rule and the strain energy density rule (also referred to as Glinka’s rule). These rules for predicting notch stresses and strains are applicable in situations where the magnitude of the nominal stress is below the material’s yield strength. If the nominal stress exceeds the yield strength, gross plastic deformation (i.e. plastic collapse) analysis may be required. This, however, is not the usual case for notched members designed against fatigue failure.

2.1 Linear rule

The linear rule is based on the assumption that the strain concentration factor is the same as the elastic stress concentration factor, \( K_t \). The notch root strain can then be expressed as 
\[
\epsilon = K_t \epsilon = K_t e.
\]

Stephens et al. [1] suggest that this rule agrees well with measurements in plane strain situations, such as for circumferential grooves in shafts in tension or bending. Gohhari-Anaraki and Hardy [2] compared the calculated strains in hollow tubes subjected to monotonic and cyclic axial loading from the linear rule with predictions from finite element analyses. They reported that strain range estimations from the linear rule provided a lower bound estimate and were up to 50 per cent less than the predictions from finite element analysis results.

2.2 Neuber’s rule

Neuber’s rule is most commonly expressed in the form
\[
K_t^2 = K_t K_o = \frac{\sigma}{\epsilon}.
\]

For nominally elastic behaviour, \( e = S/E \). When the Ramberg–Osgood equation for the stress–strain relation is combined with Neuber’s rule for nominally elastic behaviour it leads to
\[
\frac{S^2 K_t^2}{E} = \frac{\sigma^2}{E} + \sigma \left( \frac{\sigma}{K} \right)^{1/n}
\]

If the nominal stress is larger than about 0.8\( S_y \), nominal behaviour usually becomes inelastic and non-linear stress–strain relations for calculating both the nominal and the local stresses and strains are used, resulting in
\[
K_t^2 \left[ \frac{S^2}{E} + S \left( \frac{S}{K} \right)^{1/n} \right] = \frac{\sigma^2}{E} + \sigma \left( \frac{\sigma}{K} \right)^{1/n}
\]

Neuber [3] derived equation (1) for a prismatic body subjected to pure shear loading. This rule has been shown to provide accurate notch strain estimates for thin sheets and plates (e.g. plane stress) and conservative estimates for thicker, more three-dimensional parts (e.g. plane strain) [4–6]. The conservative nature of Neuber’s rule for thicker parts, as pointed out by Tipton [7], is partially explained by notch root stress multiaxiality. A multiaxial notch stress state constrains plastic flow and inhibits straining along the applied loading direction. A number of attempts have been made to account for the multiaxial stress state at the notch root. A brief discussion of these follows.

Gonyea [8] accounted for the notch root biaxiality by using the deformation theory of plasticity and a von Mises effective strain (equivalent to the distortion energy approach recommended by Neuber). Dowling et al. [9] pointed out that, if the thickness is large compared with the notch root radius, a plane strain condition prevails and the resulting biaxiality is expected to alter the cyclic stress–
strain curve for the first principal direction. To obtain the modified stress–strain curve, Hooke’s law was applied to the elastic components of strain and deformation theory of plasticity to the plastic components.

Hoffman and Seeger [10, 11] proposed an approach for notch strain estimation under multiaxial loading which requires two steps. First, a relationship between the applied load and equivalent notch stress and strain is established by an extension of Neuber’s rule to multiaxial stress states by means of replacing the involved uniaxial quantities (\(\sigma, \epsilon\) and \(K\)) by the equivalent quantities (\(\sigma_q, \epsilon_q\) and \(K_q\)) based on the von Mises (or Tresca) yield criterion. In the second step, the principal stress and strain at the notch root are related to equivalent stress and strain obtained from the first step by applying plasticity theory in combination with an assumption concerning one principal stress or strain component. This proposed method was illustrated by a round bar with a circumferential notch under tensile load and a thick-walled cylinder with a triaxially stressed notch under internal pressure. By comparison with finite element predictions, the maximum deviation observed was 30 per cent.

Gowhari-Anaraki and Hardy [2, 12] modified the Neuber rule for multiaxial states of stress by substituting either equivalent or meridional stress and strain in the Neuber equation. They found that the estimated values of equivalent and meridional total stress predicted from Neuber’s rule for both monotonic and cyclic loads deviated significantly from finite element predictions. Lee et al. [13] presented a generalized method for estimating multiaxial notch strains on the basis of the elastic notch stress solutions. The notch stresses could then be calculated by any suitable plasticity model from the results of the previous step. This method utilizes a two-surface plasticity model with the Mroz hardening equation and the associated flow rule to estimate the local notch stress and strain response. Estimated notch strains showed very good correlations with the finite element analysis predictions of notched plates under monotonic tension loading, as well as with the strain measurements of notched shafts under proportional and non-proportional alternating bending and torsion loads.

2.3 Strain energy density or Glinka’s rule
Molski and Glinka [14, 15] proposed an ‘equivalent strain energy density’ model for elastic–plastic notch–stress analysis. This method is based on the assumption that, in the case of small-scale plastic yielding near a notch tip, the plastic zone is controlled by the surrounding elastic stress field and the energy density distribution in the plastic zone is almost the same as that for a linear elastic material. It has been shown [14] that this assumption holds until general plastic yielding occurs. It has also been shown [14] that Neuber’s rule has the same energy density interpretation in the elastic regime, but the Neuber stress–strain product differs from the strain energy density for the elastic–plastic regime.

For a plane stress condition and a Ramberg–Osgood type material stress–strain behaviour, Glinka’s rule for nominally elastic behaviour is expressed as [14, 15]

\[
\frac{S^2 K^2}{E} = \frac{\sigma^2}{E} + \frac{2\sigma}{n+1} \left(\frac{\sigma}{K}\right)^{1/n}
\]  

(4)

The only difference with the Neuber rule, equation (2), is the factor \(2/(n+1)\). Since \(n < 1\), this term is larger than unity, which means that a smaller value of \(\sigma\) will satisfy the equation for a given nominal stress \(S\), compared with Neuber’s rule.

In order to satisfy the equilibrium conditions, stress redistribution occurs in the neighbourhood of the notch tip, resulting in an increase of the plastic zone size. Glinka [16] improved the calculation of the strain energy density by a factor, \(C_p\), to account for the increase in plastic zone size:

\[
C_p = 1 + \frac{\Delta r_p}{r_p}
\]  

(5)

where \(r_p\) is the plastic zone size and \(\Delta r_p\) is the increment of the plastic zone size due to the stress redistribution caused by plastic deformation. The expression for this correction factor under tension loading condition is given in reference [16]. The theoretical range of \(C_p\) values is between 1 and 2. The strain energy density rule with this correction was shown to provide good results almost up to general plastic yielding:

\[
C_p \frac{S^2 K^2}{E} = \frac{\sigma^2}{E} + \frac{2\sigma}{n+1} \left(\frac{\sigma}{K}\right)^{1/n}
\]  

(6)

Under a plane strain condition, a biaxial stress state is present at the notch tip. Therefore, the uniaxial \(\sigma–\epsilon\) curve was transformed into the biaxial \(\sigma_y^*–\epsilon_y^*\) curve by using expressions derived by Dowling et al. [9] based on plastic deformation theory:

\[
\epsilon_y^* = \frac{\sigma_y^*}{E^*} + \left(\frac{\sigma_y^*}{K^*}\right)^{1/n^*}
\]  

(7)

The \(\sigma_y^*–\epsilon_y^*\) curve accounts for the effect of \(\sigma_z\) under elastic–plastic deformation conditions. The material constants \(K^*\) and \(n^*\) have to be determined by using linear regression analysis of \(\sigma_y^*\) versus \((\epsilon_y^*)_p\) data, analogous to the determination of \(K\) and \(n\) for the uniaxial stress–strain relationship. Thus, the expression relating the nominal elastic stress in the net cross-section, \(S\), and the \(\sigma_y^*\) stress component in the plastically deformed notch tip in plane strain is
This expression also includes the plastic zone correction factor, \( C_p \). The expression for this correction factor under tension loading condition is given in reference [17]. Verifications for equation (8) were obtained by Glinka et al. [15–17] on the basis of elastic–plastic finite element analysis predictions for both monotonic and cyclic loadings.

Sharpe and co-workers [4–6] used finite element analyses and a unique laser-based technique capable of measuring biaxial strains over very short gauge lengths to evaluate the Neuber and Glinka models. Their results, as well as those from earlier studies by other researchers using foil gauges, led to the general conclusion that, for cyclic as well as monotonic loadings, Neuber’s rule works best when the local region is in a state of plane stress and Glinka’s rule is best for plane strain condition. They also suggested that it was useful to quantify the amount of notch constraint by defining this as \( \alpha = \varepsilon_2/\varepsilon_1 \), where \( \varepsilon_2 \) is the transverse strain and \( \varepsilon_1 \) is the axial strain. A value of \( \alpha = -\nu \) (Poisson’s ratio) implies a plane stress condition and \( \alpha = 0 \) implies a plane strain condition, with a value of \( \alpha = -0.2 \) to be a useful divider between ‘nearly plane stress’ and ‘nearly plane strain’ conditions. For intermediate levels of constraint which are neither plane stress nor plane strain, Sharpe and colleagues proposed [5] a modification to the Glinka rule.

Tashkinov and Filatov [18] reported an improved energy density method for inelastic notch tip strain calculations. They suggested that, by using a partial power approximation of the stress–strain curve of the material, the correction factor \( C_p \) in Glinka’s rule could be expressed explicitly. In this approximation, the stress–strain curve is represented by \( S = S_y(\varepsilon/\varepsilon_Y) \) for \( S \leq S_y \) and \( S = S_y(\varepsilon/\varepsilon_Y)^m \) for \( S > S_y \), where \( \varepsilon Y \) is the strain at yield and \( m \) is a material constant. The energy postulate of the energy density method was extended to the generalized plane strain and axisymmetric conditions by accounting for the effect of \( \sigma_2 \) on elastic–plastic deformation. A scheme for analysis was proposed for the case of nominal plastic yield. The results of the improved energy density method were compared with the finite element predictions and experimental data. Satisfactory predicted accuracy of results was reported.

### 2.4 Other stress–strain relations

As pointed out earlier, Neuber’s rule has been suggested to be suitable for plane stress situations and the linear rule for plane strain situations. An intermediate formula can be expressed as

\[
\varepsilon = K_t e \left( \frac{K_t}{K_G} \right)^m
\]

where \( m = 0 \) for plane strain (the linear rule), \( m = 1 \) for plane stress (Neuber’s rule) and \( 0 < m < 1 \) for intermediate situations. Gowhari-Anaraki and Hardy [2, 12] found that the intermediate rule \( (m = 0.5) \) was appropriate for axisymmetric components in most of the cases they studied. Sharpe and Wang [4] reported the results of biaxial notch root strain measurements on three sets of double-notched aluminium specimens that have different thicknesses and notch root radii. Elasto-plastic strains were measured with a laser-based in-plane interferometric technique. The measured strains were used to compute \( K_t \) directly and \( K_G \) using the uniaxial stress–strain curve. The exponent \( m \) in equation (9) could then be determined. The values of \( m \) were found to be 0.65, 0.48 and 0.36 for the three sets of specimens.

Ellyn and Kujawski [19] proposed a method whereby the maximum stress and strain at the notch roots could be determined for monotonic as well as cyclic loadings from the knowledge of the theoretical stress concentration factor, \( K_t \). This method is based on an averaged similarity measure of the stress and strain energy density along a smooth notch boundary. The method can also be used in the case of multiaxial states of stress. The Neuber and Glinka rules could then be derived as particular cases of the Ellyn and Kujawski method. When the nominal stress is below the yield stress, the Ellyn and Kujawski equation is the same as Glinka’s equation for a plane stress condition. Ellyn and Kujawski reported that the predicted stresses and strains at the notch root were in good agreement with the available experimental data and finite element results.

James et al. [20] proposed a simple, approximate numerical method of calculating plastic notch stresses and strains. The method ignores the compatibility condition and uses the total deformation theory of plasticity. It starts with the analytical elastic stress distribution for hyperbolic notches and predicts elastic stress and strain distributions for semicircular and U-shaped notches. In comparison with the results from a plane stress finite element analysis, the notch root strain was underestimated by 20–30 per cent. Numerical predictions of notch root conditions were found to be very close to those found using a plane stress finite element analysis.

Seshadri and Kizhatil [21, 22] proposed a generalized local stress–strain (GLOSS) plot method which could be used to predict the inelastic strains in notched components with reasonable accuracy. The GLOSS diagram is a plot of the normalized equivalent stress versus the normalized equivalent total strain that is generated from two linear elastic finite element analyses. The first finite element analysis is based on the assumption that the entire material is linear elastic. A second finite element analysis is then carried out after ‘artificially’ reducing the elastic moduli of all elements which exceed the yield stress. Therefore, the inelastic response of the local region due to plastic
3 NOTCH GEOMETRIES, MATERIAL AND STRAIN MEASUREMENTS

3.1 Notch geometries

Circumferentially notched round bar and double-notched flat plate geometries, each with different notch radii and, consequently, different stress concentration factors, $K_t$, were used. The notch configurations and dimensions are shown in Fig. 1. The notched round geometry (Fig. 1a) with a notch depth of 3.175 mm and either a notch radius of 0.529 mm or a notch radius of 1.588 mm was used to investigate the notch behaviour under plane strain condition. The double-notched plate geometry either with a notch radius and notch depth of 9.128 mm (Fig. 1b) or with a notch radius of 2.778 mm and a notch depth of 6.35 mm (Fig. 1c) was used to investigate the notch behaviour under plane stress condition.

3.2 Material

The material used in this study was an AISI 1141 medium carbon steel, microalloyed (MA) with vanadium. MA steels derive their mechanical property improvements over the conventional quenched and tempered (QT) steels from the microstructural modifications achieved by the addition of small amounts of the MA elements such as vanadium (V), niobium (Nb), titanium (Ti) and aluminium (Al). An overview on the metallurgical as well as the mechanical behaviour aspects of MA steels has been published by Yang et al. [23, 24]. Since the heat treatment process is eliminated in most MA steel productions, the desired microstructure and, therefore, properties are mainly obtained by thermomechanical processing, rather than the traditional heat treatment in QT steels. This has led to ever-increasing applications of these steels in a variety of engineering situations, particularly automotive components. Despite the relatively recent popularity of MA steels, however, investigation of their performance under cyclic loading conditions has been very limited.

MA steels typically exhibit slight cyclic softening at low strains, followed by appreciable hardening at higher strain levels. QT steels, on the other hand, cyclically soften at all strain levels, often significantly. The vanadium-based MA steel used in this investigation is a common type of MA steel and exhibits cyclic softening below 0.5 per cent strain and cyclic hardening above 0.5 per cent strain. Axial monotonic and cyclic deformation properties of the material are listed in Table 1 and include the constants used in the Ramberg–Osgood equations representing the experimental monotonic and stable cyclic stress–strain curves.

3.3 Experimental strain measurements

For notched plate specimens with the notch radius of 9.128 mm, notch root strains were measured by means of miniature electrical resistance strain gauges with an active gauge length of 0.79 mm. This gauge length is small compared with the notch dimensions shown in Fig. 1b. Finite element results indicate a nearly uniform strain contour in the axial direction over the strain gauge length. The strain gauges were carefully positioned in the loading direction at the notch root on the lateral surfaces of the specimens.

A 100 kN closed-loop servohydraulic testing machine with a digital controller was used to conduct the tests. A pair of monoball grips were used to hold the specimens in series with the load cell and loading actuator in the test machine. The specimens were subjected to pulsating axial loads with a load ratio of $R = P_{\text{min}} / P_{\text{max}} = 0.01$. A strain indicator and a switch and balance unit were used to measure strains and load versus strain data were recorded by an $x$–$y$ recorder. During monotonic as well as cyclic loadings, the applied loads were increased slowly to the next load level, to avoid transient effects. Hold times were allowed for strain stabilization, with no creep deformation observed during the hold times. To reduce any effects due to any bending stress, the measured strains from gauges positioned on the two sides of each specimen were averaged.

4 ELASTIC BEHAVIOUR AND STRESS CONCENTRATION FACTORS

4.1 Analytical methods

The elastic stress concentration factor can be estimated by three different analytical methods. The first method involves an interpolation between two exact limiting cases for deep hyperbolic notches and shallow elliptical notches [26], giving estimates which are inherently too low. The second method is based on the stress concentration factor for an elliptical hole in an infinite plate, $K_t = 1 + 2\sqrt{a/r}$, modified by a factor to correct for finite geometry. Here $a$ is the semi-axis and $r$ is the radius of curvature at the end-point of $a$. The third method makes use of the results from fracture mechanics analysis for cracked bodies and results in [27]

$$K_t = 1 + 2F \frac{\sqrt{a}}{\rho}$$

(10)

where $a$ is the notch depth, $\rho$ is the notch root radius and $F$ is a dimensionless geometry correction factor. From
fracture mechanics, $F = K/S\sqrt{\pi a}$, where $K$ is the stress intensity factor and $S$ is the nominal stress. $F$ for different crack geometries can be obtained from handbooks of stress intensity factors for cracked bodies [28, 29]. It should be noted that $K_1$ in equation (10) is defined on the basis of remotely applied nominal stress (e.g. where $K_1$ is the maximum notch root stress divided by the gross section nominal stress). The more commonly used definition of $K_1$, however, is based on the net section stress (e.g. where $K_1$ is the maximum notch root stress divided by the net section nominal stress). Conversion between the two definitions is straightforward since $(K_1S)_{\text{gross}} = (K_1S)_{\text{net}}$.

**Fig. 1** Notched configurations and dimensions used: (a) circumferentially notched round bar with 1.588 mm (1/16 in) or 0.529 mm (1/48 in) notch radii, (b) double-notched flat plate with 9.128 mm (23/64 in) notch radius and (c) double-notched flat plate with 2.778 mm (7/64 in) notch radius.
For a double-edged U-notch in a finite-width long strip with rectangular cross-section, and a circumferentially notched round bar under remote tension, the dimensionless geometry correction factors, \( F \), are given in reference [30]. This factor for the double-edged U-notch in a plate is given by

\[
F = \left[ 1 + 0.122 \cos^4 \left( \frac{\pi a}{w} \right) \right] \sqrt{\frac{w}{\pi a}} \tan \left( \frac{\pi a}{w} \right)
\]  

(11)

where \( w \) is the gross width of the plate (e.g. \( w = 41.12 \text{ mm} \) for the notched plate geometry in Fig. 1b, and \( w = 35.56 \text{ mm} \) for the notched plate geometry in Fig. 1c). Values of \( a \), \( r \) and \( F \) for the notched geometries used in this study are listed in Table 2. \( K_t \) values based on the gross section nominal stress as calculated from equation (10) were converted to \( K_t \) values based on the net section nominal stress, as described above. For example, for the notched plate geometry of Fig. 1b, \( (K_t)_{\text{net}} = (K_t)_{\text{gross}} \left( \frac{w_{\text{net}}}{w_{\text{gross}}} \right) \) with \( w_{\text{net}}/w_{\text{gross}} = 22.86/41.12 = 0.556 \). Therefore, all \( K_t \) values listed in Table 2 are based on net section nominal stress.

### 4.2 Finite element results

The finite element program used in this study was ANSYS. For the notched plate geometries under axial tensile loading, because of the symmetry in both geometry and loading, one-fourth of the plate was modelled by using two-dimensional solid plane stress elements with thickness input. For the notched bar geometries, axisymmetric two-dimensional models were employed. A far-field uniform tensile stress was applied to the end of the bar as the applied tensile loading for each calculated model. Four-node isoparametric elements were employed. About 400 nodes and elements were used in all models, with the size of the elements in the notched area progressively reduced, to trace the strain variation caused by the high gradient more accurately. The smallest elements at the notch root had an area on the order of \( 10^{-2} \text{ mm}^2 \). The accuracy of the finite element analysis models was checked by monitoring the 'strain jump' at the nodes, which is the difference between the strain values calculated for a node from each of the two adjacent mesh elements located at the notch root.

The elastic stress concentration factors, \( K_t \), from the finite element models based on the net cross-sectional area at the notch root are also listed in Table 2. The stress gradients in all cases were steep and, the smaller the notch root radius, the higher the stress gradient. Notch tip stress distributions for the notched rod and notched plate geometries with similar \( K_t \) values were very similar and do not exhibit any strong dependence on the global geometry of the notched body. This observation is in good agreement with the conclusion drawn by Glinka and Newport [31].

In order to verify the stress state in the notch region, the degree of constraint as quantified by Sharpe et al. [5] was obtained from finite element analysis results and listed in Table 3. From this table, it can be concluded that the condition at the notch root for notched round bar geometries, with \( \alpha \) being nearly zero, is very close to plane strain. For the notched flat plate geometries the notch root condition is plane stress, as expected. For this case \( \beta = 0 \) and \( \alpha \) is close to \(-\nu\), where the elastic Poisson’s ratio, \( \nu \), is 0.3. Even though the values of \( \alpha \) and \( \beta \) listed in Table 3 are for elastic loading, these values did not change significantly for inelastic loading, as discussed in Section 6.1. Therefore, the constraint states of all notch geometries investigated remain essentially unchanged for inelastic loading.

### 5 INELASTIC NOTCH BEHAVIOUR UNDER MONOTONIC LOADING

#### 5.1 Finite element and experimental results

Elastic–plastic finite element analyses were also conducted by using the ANSYS option of multilinear kinematic

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**Table 1** Mechanical properties of the material [25]

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hardness (HB)</td>
<td>262</td>
</tr>
<tr>
<td>Modulus of elasticity, ( E ) (GPa)</td>
<td>200</td>
</tr>
<tr>
<td>Yield stress (0.2 per cent), ( S_y ) (MPa)</td>
<td>524</td>
</tr>
<tr>
<td>Ultimate strength, ( S_u ) (MPa)</td>
<td>875</td>
</tr>
<tr>
<td>Reduction in area (%)</td>
<td>40.2</td>
</tr>
<tr>
<td>Strength coefficient, ( K ) (MPa)</td>
<td>1533</td>
</tr>
<tr>
<td>Strain hardening exponent, ( n )</td>
<td>0.185</td>
</tr>
<tr>
<td>Cyclic yield stress (0.2 per cent), ( S'_y ) (MPa)</td>
<td>564</td>
</tr>
<tr>
<td>Cyclic strength coefficient, ( K' ) (MPa)</td>
<td>1205</td>
</tr>
<tr>
<td>Cyclic strain-hardening exponent, ( n' )</td>
<td>0.122</td>
</tr>
<tr>
<td>Cyclic modulus of elasticity, ( E' ) (GPa)</td>
<td>200</td>
</tr>
</tbody>
</table>

**Table 2** Comparison of the \( K_t \) values for axial loading

<table>
<thead>
<tr>
<th></th>
<th>Notched round bar</th>
<th>Notched flat plate</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Case 1</td>
<td>Case 2</td>
</tr>
<tr>
<td>Notch depth, ( a ) (mm)</td>
<td>3.175</td>
<td>3.175</td>
</tr>
<tr>
<td>Notch root radius, ( r ) (mm)</td>
<td>1.588</td>
<td>0.529</td>
</tr>
<tr>
<td>Geometry correction factor, ( F )</td>
<td>1.934</td>
<td>1.934</td>
</tr>
<tr>
<td>( K_t ) based on equation (10)</td>
<td>1.59</td>
<td>2.58</td>
</tr>
<tr>
<td>( K_t ) from finite element analysis</td>
<td>1.79</td>
<td>2.83</td>
</tr>
<tr>
<td>( K_t ) from strain gauges</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
hardening. This option employs the von Mises yield criterion with the associated flow rule and kinematic hardening to compute the plastic strain increment. The finite element meshes were the same as those for linear analysis (e.g. elements at the notch root had an area on the order of $10^{-2}$ mm$^2$). The Newton–Raphson procedure in which the stiffness matrix was updated at every equilibrium interaction was used. The material properties for monotonic loading were taken from the experimental stress–strain curve. The material response was modelled by using multilinear stress–strain relations, rather than a Ramberg–Osgood type equation.

The ratio of the maximum stress at the notch root to the nominal stress, $K_\sigma$, and the ratio of the maximum strain at the notch root to the nominal strain, $K_\varepsilon$, under monotonic tension loading conditions are plotted in Fig. 2. With reference to these figures, it can be seen for round bar geometries in the elastic range that the $K_\varepsilon$ values are somewhat smaller than $K_\sigma$ values owing to the effect of the triaxial state of stress (e.g. notch constraint, since the notch

<table>
<thead>
<tr>
<th>Specimen type</th>
<th>Notch radius (mm)</th>
<th>$K_\sigma$</th>
<th>$\alpha = \varepsilon_2/\varepsilon_1$</th>
<th>$\beta = \sigma_2/\sigma_1$</th>
<th>Stress state</th>
</tr>
</thead>
<tbody>
<tr>
<td>Notched round</td>
<td>1.588</td>
<td>1.79</td>
<td>-0.057</td>
<td>0.253</td>
<td>Plane strain</td>
</tr>
<tr>
<td>Notched round</td>
<td>0.529</td>
<td>2.83</td>
<td>-0.0003</td>
<td>0.315</td>
<td>Plane strain</td>
</tr>
<tr>
<td>Notched plate</td>
<td>9.128</td>
<td>1.77</td>
<td>-0.304</td>
<td>0</td>
<td>Plane stress</td>
</tr>
<tr>
<td>Notched plate</td>
<td>2.778</td>
<td>2.75</td>
<td>-0.311</td>
<td>0</td>
<td>Plane stress</td>
</tr>
</tbody>
</table>

**Fig. 2** Variations of stress and strain concentration factors with nominal stress
is under the plane strain condition. For plate geometries, the $K_e$ values are equal to the $K_0$ values in the elastic range owing to the plane stress conditions. Above the yield point, $K_e$ increases and $K_0$ decreases, as the nominal stress is increased.

The experimental values of strains at the notch root for notched plate specimens with $K_t = 1.77$ under monotonic loading from duplicate tests are compared with the calculated results from finite element analysis in Fig. 3. As can be seen from this figure, the data obtained from experiments are very close to those from the finite element analysis, when the nominal stress is smaller than $0.8S_y$. As the nominal stress increases, the difference between measured and calculated strains also increases. At the notch strain of 0.01 this difference is 6 per cent. The measured experimental strains are lower than the finite element analysis predictions. This is expected, since strain measurements from strain gauges cannot exceed the actual strain at the notch root, because the active gauge length of 0.79 mm lies away from the notch tip.

5.2 Predictions by notch stress and strain models and comparison of results

In using Glinka’s rule for the notched round bars notch root strains were calculated by the plane strain version, whereas for notched flat plates notch root strains were calculated by using the plane stress version. The plastic zone correction factor, $C_p$, was applied to the notched geometries under plane strain condition.

Figure 4 presents and compares the calculated results of notch root strains by using finite element analysis, the linear rule, Neuber’s rule and Glinka’s rule. It is evident from this figure that, as the nominal stress increases, the differences in the calculated notch root strains from the different approaches also increase. When the nominal stress is larger than $0.8S_y$, the differences of the calculated notch root strains from finite element analysis and analytical approaches become large.

For notched round bars (plane strain state), notch root strains from the linear rule are closer to finite element analyses, compared with those from Neuber’s rule. The Neuber rule gives overly conservative results, especially at high nominal stresses. Compared with the linear and Neuber rules, notch root strains from the Glinka rule are closest to the predictions from finite element analyses for $S < 0.8S_y$. Calculated notch root stresses from the linear rule deviate significantly from finite element analysis predictions, compared with the results from other rules. Predictions of notch root stresses from the Glinka rule are closest to the predictions from finite element analysis.

For notched flat plates (plane stress state), notch root strains from the linear rule are smaller than finite element analysis predictions when the nominal stress is smaller than $0.8S_y$ and larger than FEA predictions when the nominal stress is larger than $0.8S_y$. The Neuber rule gives conservative notch root strains. Notch root stresses calculated from the Neuber rule, however, are closest to the finite element analysis predictions.

6 INELASTIC NOTCH BEHAVIOUR UNDER CYCLIC LOADING

6.1 Finite element results

The element type, yield criterion, plastic flow rule and procedure employed in analysing the inelastic cyclic notch behaviour were the same as those used for monotonic loading analysis. The material response was modelled by using multilinear cyclic stress–strain relations, based on the experimental stress–strain curve, rather than a Ramberg–Osgood equation. The finite element calculations were, therefore, monotonic but with cyclic material properties.

The variation of notch constraint index, $\alpha$, versus nominal stress amplitude for different notch geometries was examined. This index is defined as $\varepsilon_{a2}/\varepsilon_{a1}$, where $\varepsilon_{a1}$ and $\varepsilon_{a2}$ are the first and second principal strain amplitudes respectively. In the elastic range, $\alpha$ remains constant. As plastic deformation begins at the notch root, $\alpha$ changes with increased plastic deformation. For the notched plates the stress state remains plane stress and $\alpha$ gradually changes from $-0.3$ to $-0.42$, as the nominal stress amplitude increases to the cyclic yield strength of the material. For fully plastic behaviour, $\alpha$ is expected to reach $-0.5$. For the notched round bars, $\alpha$ remains nearly zero for $K_t = 2.83$ and approaches $-0.08$ for $K_t = 1.79$ at nominal stress amplitude equal to the cyclic yield strength, which is still very close to the plane strain condition. Therefore, the
constraint state at the notch root does not significantly change with plastic deformation.

### 6.2 Predictions by notch stress and strain models and comparison of results

Under cyclic loading condition, the stress and strain quantities in the analytical equations for monotonic loading are replaced by the corresponding stress and strain amplitudes. Also, material monotonic deformation properties used in these equations \((E, K, n, E^*, K^*, n^*)\) are replaced with the corresponding cyclic properties \((E', K', n', E'^*, K'^*, n'^*)\).

For notched round bars notch root strain and stress amplitudes were calculated by using the plane strain version of Glinka's rule similarly to monotonic loading, whereas for double-notched plates the plane stress version of this rule was used. The plastic zone correction factor was applied to the analysis for the plane strain condition, as was the case for monotonic loading.

Figure 5 presents the calculated notch root strain amplitudes as a function of nominal stress amplitude by using finite element analysis, the linear rule, Neuber’s rule and Glinka’s rule. It is evident from this figure that, as the nominal stress amplitude increases, so do the differences between notch root strain amplitudes from these rules.

For both notched round bars and flat plates, good agreement is observed between the results from the Glinka rule and finite element analysis predictions, if the nominal stress amplitude is below \(0.8S'_y\). Significant differences are found between the results from Glinka’s rule and Neuber’s rule for the notched round bars. For all cases, Neuber’s rule overestimates the notch root strain amplitudes, resulting in the most conservative predictions, compared with the finite element analysis predictions. The notch root strain amplitudes from the linear rule are close to finite element analysis predictions for notched plates with \(K_t = 1.77\).
However, for all cases, the linear rule results in the most non-conservative predictions of notch root stress amplitudes, compared with finite element analysis predictions.

The finite element analysis predictions agree with the conclusion that Glinka’s rule is suitable for calculating notch root strain and stress amplitudes of a notched component, where the notch is under either a plane stress or a plane strain condition. Neuber’s rule may only be suitable for calculating notch root strain and stress amplitudes of the notched component, where the notch stress state is plane stress.

7 DISCUSSION

For the finite element analysis calculations, the experimental monotonic and cyclic stress–strain curves were represented in a multilinear fashion. However, in order to obtain a closed-form solution for the analytical models evaluated, the Ramberg–Osgood model was used to idealize both monotonic and cyclic stress–strain behaviour. For the latter, the Ramberg–Osgood equation fits the experimental stress–strain curve with sufficient accuracy. For monotonic stress–strain behaviour, however, the Ramberg–Osgood equation does not represent the actual stress–strain curve very well. Therefore, for monotonic loading, one source of discrepancy between the finite element analysis predictions and predictions from the analytical models evaluated is the use of Ramberg–Osgood equation to represent the monotonic stress–strain curve.

When the nominal stress exceeds 80 per cent of the yield strength of the material, the nominal behaviour becomes inelastic. In this case, the linear rule, Neuber’s rule and Glinka’s rule are modified for the non-linear nominal stress–strain relation. Therefore, the curve shifts observed

Fig. 5 Notch root strain amplitudes from finite element analysis, the linear rule, the Neuber rule and the Glinka rule under cyclic axial loading.
at about 80 per cent of the yield strength in Figs 4 and 5, based on these models, are caused by switching from linear nominal stress–strain behaviour for $S < 0.8S_y$ to non-linear nominal behaviour for $S > 0.8S_y$.

Stable notch root strain amplitudes were measured by miniature strain gauges using notched flat plate specimens with larger notch root radii, subjected to pulsating cyclic loading. The measured stable strain amplitudes compared very well with the finite element analysis predictions. However, the experimental results obtained were mainly in the linear and relatively small inelastic notch behaviour region owing to the short fatigue life of strain gauges at larger strain amplitudes. The laser-based interferometric technique developed by Sharpe [32] is very valuable for experimental investigation of notch cyclic strain behaviour with significant cyclic plasticity and/or for steep or small notches requiring a very short gauge length.

For finite element analysis, many factors such as the type and size of elements used and the type of elasto-plastic model can significantly influence the results obtained. In addition, such analysis assumes continuum elasto-plasticity, which may not be fully met for some materials and/or conditions (e.g., if the grain size is not very small, compared with the notch dimensions). Therefore, the finite element analysis predictions may not necessarily be the most accurate. However, because of the experimental difficulties in using resistance strain gauges mentioned above, most investigators have evaluated notch deformation models based on finite element analyses.

On the basis of the notch root stress and strain results presented, large differences can be observed between the analytical models. This will result in even larger differences in predicted fatigue lives of notched components, since relatively small variations in notch root stress or strain amplitude can result in significant differences in predicted lives.

Evaluation of the analytical rules evaluated by comparisons with the finite element analysis predictions indicate that these rules generally underpredict notch root stress and overpredict notch root strain for both notched round bars and flat plates and under both monotonic and cyclic loading conditions. Notch root stress and strain predictions from each rule were mainly consistent between the two notch geometries and for both monotonic and cyclic loadings. In fatigue design of notched members based on the local approach, one method often used to reduce the degree of conservatism in the Neuber rule is to replace $K_t$ with $K_I$, which is the fatigue notch factor [1]. This factor is smaller than $K_t$, and, therefore, its use results in lower predicted notch root stress and strain. The strain energy density (Glinka’s rule) gives the best overall notch root stress and strain predictions, as compared with the predictions from finite element analyses, for both notch geometries (plane stress and plane strain) and under both monotonic and cyclic loads in this study. However, a high degree of accuracy in notch root stress and strain predictions should not generally be expected from any of the analytical models. These models should only be used as first estimates, and a more accurate prediction would require detailed computational analysis and/or careful experimental measurements. Which approach to use depends on the safety factor required based on the safety-critical nature of the part and, as is often the case, by weighting the use of a less accurate but simpler and less expensive analytical model with either a more accurate but also more time-consuming and more expensive computational model or an experimental approach.

Differences between predictions from finite element analysis and the notch root stress and strain models often become very large after the nominal stress exceeds the yield strength. For engineering load-carrying members, however, the nominal stress is usually smaller than the yield strength, even though the local stress exceeds the yield strength. In situations where the nominal stress exceeds the yield strength, gross plastic deformation analysis would be required, as previously discussed.

It is worth mentioning that monotonic and cyclic deformation of the material in the QT condition was also investigated, but the results obtained are not included to avoid duplication. The QT treatment of the as-forged (AF) condition was done in a manner to produce an equivalent hardness to the AF condition (e.g., 260 HB). This QT treatment increased the yield strength from 524 MPa for the AF condition to 670 MPa and decreased the ultimate tensile strength from 875 MPa for the AF condition to 777 MPa. The material in the QT condition cyclically softened at all strain levels. Investigation of the QT condition was conducted for the notched round bar geometry with $K_t = 1.79$ and the notched flat plate geometry with $K_t = 1.77$ by using the analytical models (linear rule, Neuber’s rule and Glinka’s rule), elastic as well as elasto-plastic finite element analyses, and experimental measurements of notch root strains for the notched plate geometry. However, all the results obtained were analogous to those for the AF material condition and the conclusions reached regarding experimental, finite element analysis and analytical model results were also the same as those for the AF material.

8 SUMMARY

The notch root stress and strain behaviour of a vanadium-based MA steel under both monotonic and cyclic loading conditions was investigated using circumferentially notched round bars and double-notched flat plates. The stress state at the notch root for the notched round bars is plane strain, whereas for the notched flat plates it is plane stress. The stress concentration factors were calculated and compared with the experimental and finite element analysis results.

Experimental values of strains at the notch root for notched plate specimens with $K_t = 1.77$ under monotonic
loading were in close agreement with the calculated results from finite element analyses when the nominal stress was smaller than 0.8S.<. As the nominal stress increased, the difference between the measured and finite element analysis calculated strains also increased. However, this difference was still only 6 per cent for a notch strain level of 0.01. Stable notch root strain amplitudes measured by strain gauges for the same plate specimens subjected to pulsating cyclic loading were also in close agreement with the finite element analysis predictions. However, the experimental results obtained for cyclic loading were limited owing to the short fatigue life of strain gauges at large strain amplitudes.

Notch root stresses and strains were calculated by employing the linear rule, Neuber’s rule and Glinka’s rule under both monotonic and cyclic loads and were compared with elastic–plastic finite element analysis predictions. The Neuber rule predicted conservative local strain amplitudes, especially when the local stress state is plane strain. The results from the Glinka rule were closest to the predictions from finite element analyses under both plane stress and plane strain conditions and for both monotonic and cyclic loading conditions.

REFERENCES


